Decoupling limit and AdS/CFT

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We have seen that the effective action of massless string modes in the presence of a D3-brane takes the form

$$I = I_{bulk} + I_{brane} + I_{int}$$

where $I_{bulk} = \frac{1}{16\pi G} \int_{bulk} d^{10}x \sqrt{-g}\, e^{-2\Phi}\left(R + 4(\nabla\Phi)^2 + \dots\right)$: IIB sugra + corrections

$I_{brane} = \frac{1}{2g_{YM}^2} \int_{WV} d^4x \sqrt{-h}\, \mathrm{Tr}(F^2) + \dots$: SYM + corrections

and I_{int} includes the interactions between open and closed string modes.

We have $16\pi G \sim g_s^2 l_s^8$ and $g_{YM}^2 \sim g_s$

Now consider a low energy limit, with energies $<< m_s$. Keep the energy and g_s fixed and send $m_s \to \infty$, ie $l_s \to 0$, so $G \to 0$

Then I_{int} vanishes, and I_{bulk} becomes a free theory. The YM theory remains. We have decoupled the free gravity in the bulk and the gauge theory on the brane.

Now look at this limit from the point of view of the D3-brane solution of supergravity.

$$ds^2 = h^{-1/2}(r)\left(-dt^2 + d\vec{x}_3^{\,2}\right) + h^{1/2}(r)\left(dr^2 + r^2 d\Omega_5\right)$$

$$h(r) = 1 + \frac{4\pi g_s \ell_s^4 N}{r^4}$$

$$F_{(5)} = (1+*)\, dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\frac{1}{h}$$

There is a redshift from $-g_{tt} = \frac{1}{h^{1/2}}$ which grows as $r \to 0$: This is a region where excitations have small energy measured relative to $r \sim \infty$. (near a black hole, time runs slow: Interstellar)

Consider an excitation described by a massless scalar field φ in the D3 geometry (it can be, eg, a graviton component $h_{x^1x^2}$)

$$\Box \varphi = 0 \qquad \varphi = e^{-i\omega t}\varphi_\omega(r)$$

(for simplicity no dependence on $\vec{x}$)

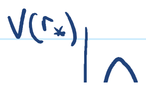
With a little manipulation the equation can be written as a Schrödinger equation:

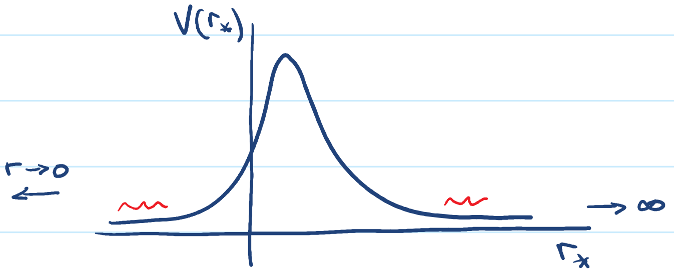
$$\left[-\frac{d^2}{dr_*^2} + V(r_*)\right]\psi(r_*) = \omega^2 \psi(r_*)$$

with $dr_* = h^{1/2}(r)\, dr$ ("Tortoise coordinate")

$r \to 0 \Rightarrow r_* \to -\infty$

and the potential





Low-energy excitations on both sides of the barrier decouple: The near-brane region decouples from weak gravity far from the brane.

Focus on the near-brane region $r \simeq 0$.

Then $$h \simeq \frac{4\pi g_s N \ell_s^4}{r^4} \equiv \frac{q^4}{r^4} \qquad q \equiv (4\pi g_s N)^{1/4} \ell_s$$

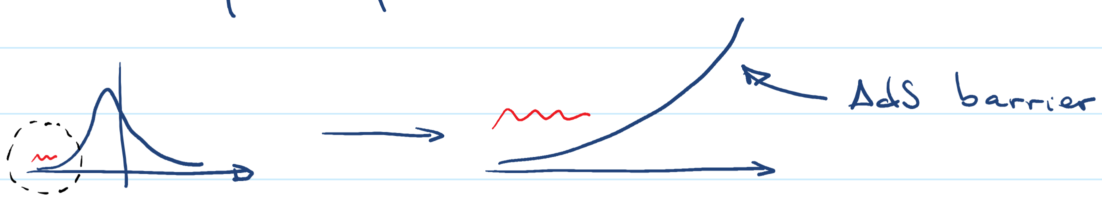
$$ds^2 \to \frac{r^2}{q^2}\left(-dt^2 + d\vec{x}_3^{\,2}\right) + \frac{q^2}{r^2}\left(dr^2 + r^2 d\Omega_5\right)$$

$$= \underbrace{q^2 \frac{-dt^2 + d\vec{x}_3^{\,2} + dz^2}{z^2}}_{} + \underbrace{q^2 d\Omega_5}_{} \qquad z = \frac{q^2}{r}$$

$$\boxed{\text{Poincaré-}AdS_5 \quad \times \quad S^5}$$

both with radius $q = L_{AdS_5} = L_{S^5}$

By rescaling, we're taking finite-energy excitations in an infinite potential well:



and we decouple this region from the far zone.

So both ways to take the low energy limit result in the decoupling of the weakly-gravitating region

So both ways to take the low energy limit result in the decoupling of the weakly-gravitating region far from the brane.

The AdS / CFT correspondence is the statement that the gravitating region in the throat near $r=0$ is equivalent to the YM theory that describes the D3-brane dynamics, in the limit where $N \to \infty$ and the coupling $g_s N \to \infty$.

Let's see these last points:

We have $g_s N \sim \frac{q^4}{\ell_s^4} = \left(\frac{L_{AdS}}{\ell_s}\right)^4 \gg 1$

Since $g_{YM}^2 \sim g_s$, then the 'tHooft coupling $\lambda = g_{YM}^2 N$ of the YM theory is large:

$$\boxed{\lambda = g_{YM}^2 N \sim \left(\frac{L_{AdS}}{\ell_s}\right)^4 \gg 1}$$

Now, since $G \sim \ell_P^8 \sim g_s^2 \ell_s^8$, we have

$$\ell_P^8 \sim g_s^2 \ell_s^8 \sim g_s^2 \frac{L_{AdS}^8}{g_s^2 N^2} = \frac{L_{AdS}^8}{N^2}$$

so $$\boxed{N \sim \left(\frac{L_{AdS}}{\ell_P}\right)^4 \gg 1}$$

So in the regime where the sugra description of the D3 brane near-horizon is valid, the YM

Theory is at large $N \gg 1$
and strong coupling $g_{YM}^2 N \gg 1$

Conversely, when $g_{YM}^2 N$ is small and we can treat the YM perturbatively, the gravitational theory has very large string effects since $l_s > L_{AdS}$

If $N \sim O(1)$ then quantum gravity effects are large

($N \gg 1$ but $g_s N$ small is classical strings)

So we have

non-perturbative

$\uparrow$

$\mathcal{N}=4$ $SU(N)$ SYM $\rightleftarrows$ Type IIB string theory on $AdS_5 \times S^5$

$\downarrow$

perturbatively in g_s

Let's take the near-horizon decoupling limit for the black D3-brane:

$$ds^2 = h^{-1/2}(r)\left(-f(r)\,dt^2 + d\vec{x}_3^{\,2}\right) + h^{1/2}(r)\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5\right)$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^4$$

$$h(r) = 1 + \left(\frac{q}{r}\right)^4 \longrightarrow \left(\frac{q}{r}\right)^4$$

make $r, r_0 \ll q$ with r/r_0 fixed

Then

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$$ds^2 \to \underbrace{\frac{r^2}{q^2}\left(-f(r)\,dt^2 + d\vec{x}_3^{\,2}\right) + \frac{q^2}{r^2}\frac{dr^2}{f(r)}}_{\substack{\text{black brane in } AdS_5 \\ \text{planar } (k=0) \text{ black hole in } AdS_5}} + \underbrace{q^2\, d\Omega_5^2}_{S^5}$$

Interesting fact: we can't obtain (don't know how to)

$(\text{global } AdS_5) \times S^5$

as the near-horizon limit of any D-brane system

but we can obtain $(\text{global } AdS_3) \times S^3$
and $(\text{global } AdS_2) \times S^2$

as near-horizon limits of: D1-D5 supertube
Traversable wormhole